Numerical modeling of bending of micropolar plates

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A B S T R A C T

In this paper we present the Finite Element modeling of the bending of micropolar elastic plates. Based on our recently published enhanced mathematical model for Cosserat plate bending, we present the micropolar plate field equations as an elliptic system of nine differential equations in terms of the kinematic variables. The system includes an optimal value of the splitting parameter, which is the minimizer of the micropolar plate stress energy. We present the efficient algorithm for the estimation of the optimal value of this parameter and discuss the approximations of stress and couple stress components. The numerical algorithm also includes the method for finding the unique solution of the micropolar plate field equations corresponding to the optimal value of the splitting parameter. We provide the results of the numerical modeling for the plates made of polyurethane foam used in structural insulated panels. The comparison of the numerical values of the vertical deflection for the square plate made of dense polyurethane foam with the analytical solution of the three-dimensional micropolar elasticity confirms the high order of approximation of the three-dimensional (exact) solution. The size effect of micropolar plate theory predicts that plates made of smaller thickness will be more rigid than would be expected on the basis of the Reissner plate theory. We present the numerical results for plates of different shapes, including shapes with rectangular holes, under different loads.

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1. Introduction

In the classical theory of elasticity the material particles of a continuum body are assumed to have three degrees of freedom, which are responsible for their macrodisplacement within the body. The size effects of the particles and their mutual rotational interactions are ignored. The surface loads are assumed to be completely determined by the force vector, which results in the asymmetry of the stress tensor and the introduction of the couple stress tensor. Such important engineering materials as chopped fiber composites, platelet composites, particulate composites, sandwich and grid structures, trusses and honeycombs are just a few examples of the materials that exhibit nonclassical behavior. Rocks, soils, concrete with sand, ferroelectric and phononic crystals, polyfoams, human bones and some biological tissues are also considered to be micropolar materials [24,18,5,19,22,27,21,17].

The development of the micropolar elasticity can be traced back to 1887 when Voigt made an assumption that the surface loads should be described by both force and moment vectors, thus introducing the concept of couple stress. In 1909 Cosserat brothers presented the equations of local balance of momenta for stress and couple stress, and the expressions for surface tractions and couples [7]. Eringen developed micromorphic and micropolar theories of solids, fluids, memory-dependent media, microstretch solids and fluids and solved several problems in these fields [11]. Eringen also introduced a theory of plates in the framework of the linear micropolar elasticity [13]. The theory was based on a technique of integration of the micropolar elasticity and assumes no transverse variation of micropolar rotations. More remarks on the history of the modeling of classic linear elastic plates can be found in [23,30,31].

The micropolar plate theory based on the Reissner plate theory was first proposed in [32] and in the final form has been developed in [33]. The preliminary computations showed that
the precision of the micropolar plate theory [33] is compatible with the precision of the Reissner elastic plate theory [10].

In this paper we present the Finite Element analysis of micropolar plate bending. In Section 2 we make an overview of the micropolar plate theory, its assumptions, expression for micropolar plate stress energy and the bending system of equations. We also present the field equations in terms of the kinematic variables and derive the explicit expression for the optimal value of the splitting parameter, which minimizes the micropolar plate stress energy. In Section 3 we provide the numerical analysis of the bending of micropolar plates. We describe the Finite Element method used to solve the micropolar field equations and discuss the approximations of stress and couple stress components. We also provide the efficient algorithm for the optimal value of the splitting parameter. The comparison of the displacement for the square plate made of dense polyurethane foam with the analytical solution of the equation for the optimal value of the splitting parameter. The algorithm for the optimal value of the splitting parameter. The obtained system. We will obtain the explicit expression for the optimal value of the splitting parameter for the ellipticity of the obtained system. We will also describe the Finite Element analysis of the plates of different construction shapes and plates with rectangular holes, under different loads.

2. Enhanced model for Cosserat plate bending

In this section we will reproduce the main micropolar plate equations and the general results of the micropolar plate theory presented in [33]. We will develop the micropolar plate field equations in terms of the kinematic variables and discuss the ellipticity of the obtained system. We will obtain the explicit expression for the optimal value of the splitting parameter for plates of arbitrary shape.

2.1. Micropolar plate equations

The equations of balance of momentum and moment of momentum of the linear theory of micropolar elasticity are respectively given by

$$\sigma_{ij} + \rho(\dot{u}_i - \ddot{u}_i/\alpha t^2) = 0,$$

$$\epsilon_{ijk} \sigma_{jk} + \mu \dot{\gamma}_{ij} + \rho(\dot{h}_i - \ddot{h}_i/\alpha t^2) = 0,$$

where $\sigma_{ij}$ is the stress tensor, $\mu_{ij}$ is the couple stress tensor, $\epsilon_{ijk}$ is the Levi-Civita symbol, $\rho$ is the mass density, $\dot{h}$ is the body force, $u_i$ is the displacement vector, $h_i$ is the body couple, $j$ is the micro-inertia and $\dot{\gamma}_{ij}$ is the microrotation vector. In this theory $\rho$ and $\alpha$ are assumed to be constant [13, 12].

For static case the equilibrium equations of the linear micropolar theory without body forces and body moments have the following form:

$$\sigma_{ij} = 0,$$

(1)

$$\epsilon_{ijk} \sigma_{jk} + \mu \dot{\gamma}_{ij} = 0,$$

(2)

The constitutive equations of the linear theory are given in the form [26]:

$$\sigma_{ij} = (\mu + \alpha) \gamma_{ij} + (\mu - \alpha) \gamma_{ij} + \lambda \delta_{ij} \gamma_{kk},$$

$$\mu \dot{\gamma}_{ij} = (\gamma + \epsilon) \epsilon_{ijk} \dot{\gamma}_{jk} + (\gamma - \epsilon) \dot{\gamma}_{ij} + \beta \delta_{ij} \dot{\gamma}_{kk},$$

(3)

(4)

where $\gamma_{ij}$ and $\dot{\gamma}_{ij}$ are the micropolar strain and torsion tensors respectively, $\mu$ and $\lambda$ are the Lamé parameters, $\alpha, \beta, \gamma$ and $\epsilon$ are the asymmetric elasticity constants.

The strain-displacement and torsion-rotation relations are given as in [12]:

$$\gamma_{ij} = U_{ij}^* - \epsilon_{ijk} \phi_{k},$$

(5)

$$\dot{\gamma}_{ij} = \dot{\phi}_{ij}.$$  

(6)

Let us consider the body $B_0$ to be a plate of thickness $h$ and $x_3 = 0$ containing its middle plane $P_0$. The sets $T$ and $B$ are the top and bottom surfaces contained in the planes $x_3 = h/2$ and $x_3 = -h/2$ respectively and the curve $I$ is the boundary of the middle plane of the plate. We consider the vertical load and the boundary conditions at the top and bottom of the plate:

$$\sigma_{33}(x_1, x_2, h/2) = \sigma^a(x_1, x_2), \quad \sigma_{33}(x_1, x_2, -h/2) = \sigma^b(x_1, x_2),$$

(7)

$$\sigma_{33}(x_1, x_2, \pm h/2) = 0,$$

(8)

$$\mu_{33}(x_1, x_2, h/2) = \mu^a(x_1, x_2), \quad \mu_{33}(x_1, x_2, -h/2) = \mu^b(x_1, x_2),$$

(9)

$$\mu_{33}(x_1, x_2, \pm h/2) = 0,$$

(10)

where $\sigma^a(x_1, x_2) \in P_0$.

We briefly review our approach presented in [33]. As it is assumed in the standard theory of plates, we use the following approximation for the stress components:

$$\sigma_{ij} = \frac{h}{2} m_{ij}(x_1, x_2),$$

(11)

where $\zeta = (2/h)x_3,$ and $\alpha, \beta \in [1, 2]$. Based on (11) and by means of the first two equations of stress equilibrium (1) written in the component form

$$\sigma_{ij} = 0$$

we obtain for the shear stress components

$$\sigma_{ij} = q_{ij}(x_1, x_2)(1-z^2)^z,$$

(12)

We use expression for the stress components [33]:

$$\sigma_{ij} = q_{ij}(x_1, x_2)(1-z^2)^z + \dot{\phi}_{ij}(x_1, x_2).$$

(13)

Substituting Eq. (13) in the remaining equilibrium differential equation for stress

$$\sigma_{ij} = 0$$

we obtain the expression for the transverse normal stress

$$\sigma_{33} = \zeta (z^2 - 1) k(x_1, x_2) + \zeta'(x_1, x_2) + m'(x_1, x_2),$$

(14)

The next step is to accommodate approximations (14) to the boundary (7). By direct substitution to (7) it easy to obtain that

$$\sigma_{33} = -\frac{3}{4} (1-z^2) k(x_1, x_2) + \zeta p_1(x_1, x_2) + \zeta p_2(x_1, x_2) + m(x_1, x_2),$$

(15)

where

$$p_1(x_1, x_2) + 2 p_2(x_1, x_2) = p(x_1, x_2)$$

We consider the parametric solution of the last equation in the form:

$$p_1(x_1, x_2) = \eta p(x_1, x_2),$$

$$p_2(x_1, x_2) = \frac{(1-\eta)}{2} p(x_1, x_2)$$

and $\eta \in [0, 1]$ is a parameter, which we call the splitting parameter. This allows us to split the bending pressure on the plate $p(x_1, x_2)$ into two parts corresponding to different orders of stress approximations. The optimal value of the splitting parameter shows the contribution of different types of approximation in the plate bending. This approach gives a more accurate description of the mechanical phenomenon of bending.
Note that for
\[ p(x_1, x_2) = \sigma^t(x_1, x_2) - \sigma^b(x_1, x_2) \]
and
\[ \sigma_0(x_1, x_2) = \frac{1}{2} \sigma^t(x_1, x_2) + \sigma^b(x_1, x_2) \]
the expression (15) satisfies the boundary condition requirements. Note that in the case of \( \eta = 1 \) expression (15) is identical to the expression of \( \sigma_{33} \) given in (32).

We use the following approximation for the couple stress components [33]:
\[ \mu_{ij} = (1 - \zeta^2)\tau_{ij}(x_1, x_2) + \tau^b_{ij}(x_1, x_2). \]
and substituting the couple stress \( \mu_{ij} \) into equation (16) and taking into account (12) and (13) we obtain the expression for the transverse shear couple stress
\[ \mu_{3b} = (\frac{1}{2} \zeta^2 - \zeta)\phi_b(x_1, x_2). \]
Substituting (19) into boundary (8) we obtain that
\[ s_b(x_1, x_2) = 0, \]
i.e. the transverse shear couple stress [32]
\[ \mu_{3b} = 0. \]
We use the assumption from [32], i.e. \( \mu_{33} \) is a first order polynomial
\[ \mu_{33} = \zeta^b(x_1, x_2) + C^b(x_1, x_2). \]
which satisfies the remaining differential equation of the equilibrium of angular momentum (2)
\[ \epsilon_{2b}e_{jk}^t + \mu_{2j} = 0. \]
This assumption is also consistent with the equilibrium equation (22) and allows us to proceed as we did for the determination of transverse loading stress (15) from the stress boundary conditions. The boundary conditions (9) are sufficient to determine \( \mu_{33} \), which must be of the form [32]
\[ \mu_{33} = \zeta + t, \]
where the function \( v(x_1, x_2) \) and \( t(x_1, x_2) \) are given as
\[ v(x_1, x_2) = \frac{1}{2} \mu^t(x_1, x_2) - \mu^b(x_1, x_2), \]
\[ t(x_1, x_2) = \frac{1}{2} \mu^t(x_1, x_2) + \mu^b(x_1, x_2). \]
The choice of kinematic assumptions is based on simplicity and their compatibility with the constitutive relationships of stress and couple stress assumptions are the following [33]:
\[ u_a = \frac{h}{2} \zeta \psi_a(x_1, x_2), \]
\[ u_3 = w(x_1, x_2) + (1 - \zeta^2)w^*(x_1, x_2). \]
The terms \( \psi_a(x_1, x_2) \) in (24) represent the rotations in the middle plane.
We also use the microrotation \( \phi_a \) in the following form [33]:
\[ \phi_a = \frac{5}{4} \Omega_0^a(x_1, x_2)(1 - \zeta^2) + \dot{\phi}_a(x_1, x_2), \]
and substituting the couple stress \( \mu_{ij} \)
\[ \sigma_{ij} = \frac{h}{2} \zeta \sigma_{ij}(x_1, x_2). \]
The constitutive formulas motivate us to chose the forms (26) and (27), which produce expressions for \( \phi_a^t \) and \( \phi_3, \dot{\phi}_3, \dot{\phi}_a \) similar to what we have for couple stress approximations (16).
Thus, we summarize the resulting assumptions for stress \( \sigma_{ij} \) and couple stress \( \mu_{ij} \) components across the thickness [33]
\[ \sigma_{ij} = \frac{h}{2} \zeta \sigma_{ij}(x_1, x_2). \]
(28)
\[ \sigma_{3j} = \dot{\phi}_j(x_1, x_2)(1 - \zeta^2), \]
(29)
\[ \sigma_{ij} = \dot{\phi}_j(x_1, x_2)(1 - \zeta^2) + \dot{\phi}_j(x_1, x_2), \]
(30)
\[ \sigma_{33} = \frac{3}{4} \left( \zeta^2 - \zeta \right) p_1(x_1, x_2) + \zeta p_2(x_1, x_2) + \sigma_0(x_1, x_2), \]
(31)
\[ \mu_{ij} = (1 - \zeta^2)\tau_{ij}(x_1, x_2) + \tau^b_{ij}(x_1, x_2). \]
(32)
\[ \mu_{3j} = \frac{h}{2} \zeta \phi_j(x_1, x_2), \]
(33)
\[ \mu_{3j} = \frac{1}{2} \left( \zeta^2 - \zeta \right) s_b(x_1, x_2), \]
(34)
\[ \mu_{3j} = 0, \]
(35)
\[ \mu_{33} = \zeta + t, \]
(36)
where
\[ p(x_1, x_2) = \sigma^t(x_1, x_2) - \sigma^b(x_1, x_2), \]
\[ \sigma_0(x_1, x_2) = \frac{1}{2} \left( \sigma^t(x_1, x_2) + \sigma^b(x_1, x_2) \right), \]
\[ p_1(x_1, x_2) = \sigma t(x_1, x_2), \]
\[ p_2(x_1, x_2) = \frac{1}{2} \left( \mu^t(x_1, x_2) + \mu^b(x_1, x_2) \right), \]
\[ v(x_1, x_2) = \frac{1}{2} \mu^t(x_1, x_2) - \mu^b(x_1, x_2), \]
\[ t(x_1, x_2) = \frac{1}{2} \mu^t(x_1, x_2) + \mu^b(x_1, x_2). \]
The kinematic assumptions are given as
\[ u_a = \frac{h}{2} \zeta \psi_a(x_1, x_2), \]
(37)
\[ u_3 = w(x_1, x_2) + (1 - \zeta^2)w^*(x_1, x_2). \]
(38)
\[ \varphi_a = \frac{5}{4} \Omega_0^a(x_1, x_2)(1 - \zeta^2) + \dot{\phi}_a(x_1, x_2), \]
(39)
\[ \varphi_3 = \frac{5h}{8} \zeta \Omega_3^a(x_1, x_2). \]
(40)
We also introduce the set of kinematic variables defined as
\[ U = [u_a, v, \Omega, \varphi_a]^T, \]
and the micropolar plate stress set
\[ S = [M_{ij}, Q_{ij}, Q_{ij}^a, \dot{Q}_{ij}, \dot{Q}_{ij}^a, \dot{S}_b]^T, \]
where
\[ M_{ij} \]
\[ Q_{ij} \]
\[ Q_{ij}^a \]
\[ \dot{Q}_{ij} \]
\[ \dot{Q}_{ij}^a \]
\[ \dot{S}_b \]
\[ \dot{\mathbf{Q}} = \frac{h}{2} \int \dot{\mathbf{q}}_1 (1 - \mathbf{q}^2) \, d\zeta_3 = \frac{2h}{3} \mathbf{q}_\alpha, \quad (44) \]

\[ \dot{\mathbf{\tau}} = \frac{h}{2} \int \dot{\mathbf{r}}_\alpha (1 - \mathbf{q}^2) \, d\zeta_3 = \frac{2h}{3} \mathbf{r}_\alpha, \quad (45) \]

\[ \dot{\mathbf{S}}^\alpha = \left( \frac{h}{2} \right)^2 \int \dot{\mathbf{\tau}}_\alpha (1 - \mathbf{q}^2) \, d\zeta_3 = \frac{h^3}{12} \mathbf{S}^\alpha, \quad (46) \]

The components of the micropolar plate strain set \( \mathbf{E}^{\text{def}} \) are defined as [33]

\[ \mathbf{c}_{\text{def}} = \frac{1}{2} \int \mathbf{c}_{\alpha} \, d\zeta_3, \quad (48) \]

\[ \alpha_{\text{def}} = \frac{1}{4} \int \mathbf{r}_{\alpha} (1 - \mathbf{q}^2) \, d\zeta_3, \quad (50) \]

\[ \dot{\mathbf{\omega}} = \frac{3}{4} \int \mathbf{r}_{\alpha} (1 - \mathbf{q}^2) \, d\zeta_3, \quad (51) \]

\[ \tau_{3\alpha} = \frac{3}{h} \int \mathbf{\tau}_{3\alpha} \, d\zeta_3, \quad (53) \]

\[ \tau_{\alpha \beta} = \frac{3}{4} \int \mathbf{\tau}_{\alpha \beta} (1 - \mathbf{q}^2) \, d\zeta_3. \quad (54) \]

\[ \mathbf{W}^{\text{def}} = \frac{h}{2} \int \left( \mathbf{\sigma} \cdot \mathbf{\gamma} + \mu \cdot \mathbf{\chi} \right) \, d\zeta_3, \quad (55) \]

which is equal to the micropolar plate stress energy density [33].

The expression for the density of the work is obtained by substituting the stress and couple stress assumptions (28)–(36) and performing the integration along the thickness of the plate [33]

\[ \mathbf{W}^{\text{def}} = \mathbf{S}^{\text{def}} \cdot \mathbf{E}^{\text{def}} = \mathbf{M}_{\text{def}} \mathbf{c}_{\text{def}} + \mathbf{Q}^{\text{def}} \mathbf{\omega}_{\text{def}}, \]

\[ + \mathbf{Q}^{\text{def}} \mathbf{c}_{\text{def}} + \mathbf{Q}^{\text{def}} \mathbf{\omega}_{\text{def}} + \mathbf{R}_{\text{def}} \mathbf{\tau}_{\text{def}} + \mathbf{R}_{\text{def}} \mathbf{\tau}_{\text{def}} + \mathbf{S}^{\text{def}} \mathbf{\tau}_{3\alpha}. \quad (56) \]

The constitutive formulas are given in the following reverse form [2]:

\[ M_{\text{def}} = \frac{h^3 \mu (\lambda + \mu)}{3(\lambda + 2\mu)} \mathbf{\psi}_{\alpha \alpha} + \frac{\lambda h^3}{6(\lambda + 2\mu)} \mathbf{\psi}_{\beta \beta} + \frac{(3p_1 + 5p_2) \eta h^2}{3(\lambda + 2\mu)}, \quad (71) \]

\[ R_{\text{def}} = \frac{5(\mu - \eta) h^3}{6} \mathbf{\psi}_{\alpha \beta} + \frac{5h^3 (\lambda + \mu) h^3}{6} \mathbf{\psi}_{\alpha \beta}, \quad (73) \]

\[ M_{\text{def}} = \frac{(\lambda - \mu) h^3}{12} \mathbf{\psi}_{\alpha \beta} + \frac{h^3 (\lambda + \mu)}{12} \mathbf{\psi}_{\beta \beta} + \frac{(3p_1 + 5p_2) \eta h^2}{3(\lambda + 2\mu)}, \quad (72) \]

\[ R_{\text{def}} = \frac{5(\mu - \eta) h^3}{6} \mathbf{\psi}_{\alpha \beta} + \frac{5h^3 (\lambda + \mu) h^3}{6} \mathbf{\psi}_{\alpha \beta}, \quad (74) \]

\[ V = \frac{2h - 3\eta h^2}{3} \mathbf{\omega}_{\alpha \beta} + \frac{2h - 3\eta h^2}{3} \mathbf{\omega}_{\alpha \beta} + \frac{4h^2 \eta}{3(\lambda + 2\mu)} \mathbf{\omega}_{\beta \beta}, \quad (75) \]

\[ V = \frac{2h - 3\eta h^2}{3} \mathbf{\omega}_{\alpha \beta} + \frac{2h - 3\eta h^2}{3} \mathbf{\omega}_{\alpha \beta} + \frac{4h^2 \eta}{3(\lambda + 2\mu)} \mathbf{\omega}_{\beta \beta}, \quad (76) \]

1 Eqs. (57) and (58) are the correct versions of the equations developed in [33].

2 In the following formulas a subindex \( \beta = 1 \) iff \( \alpha = 2 \) and \( \beta = 2 \) iff \( \alpha = 1 \).
\[
\begin{align*}
Q_u &= \frac{5(h+\mu)}{6} \nu_u - \frac{5(\mu-\alpha)h}{6} W_u + \frac{2(\mu-\alpha)h}{3} W_u^*, \\
\left(1\right)^{\alpha} \frac{5h\alpha}{3} (\dot{\alpha}_p + \dot{Q}_p), \\
Q_v &= \frac{5(\mu-\alpha)h}{6} \nu_v + \frac{5(\mu-\alpha)^2 h}{6(\mu+\alpha)} W_u + \frac{2(\mu+\alpha)h}{3} W_u^*, \\
\left(1\right)^{\alpha} \frac{5h\alpha}{3} (\dot{\alpha}_p + \mu + \alpha) \omega_p, \\
Q_z &= \frac{8\alpha h}{3(\mu + \alpha)} W_u + \left(1\right)^{\alpha} \frac{8\alpha h}{3(\mu + \alpha)} \dot{\omega}_p, \\
S_{\alpha} &= \frac{5\nu h^3}{3(\gamma + e)} \omega_3. 
\end{align*}
\]

2.2. Micropolar plate field equations

In order to obtain the micropolar plate bending field equations in terms of the kinematic variables, we substitute the constitutive formulas in the reverse form (71)-(80) into the bending system of Eqs. (65)-(70). The micropolar plate bending field equations can be written in the following form:

\[\mathcal{U} = \mathcal{F}(\mathcal{T})\]  

(82)

where

\[\begin{bmatrix}
L_{11} & L_{12} & L_{13} & L_{14} & 0 & L_{16} & k L_{13} & 0 & L_{18} \\
L_{12} & L_{22} & L_{23} & L_{24} & L_{16} & 0 & k L_{23} & L_{16} & 0 \\
L_{13} & L_{23} & L_{33} & 0 & L_{35} & L_{36} & L_{77} & L_{38} & L_{39} \\
L_{14} & L_{24} & 0 & L_{44} & 0 & 0 & 0 & 0 & 0 \\
L_1 &= 0 & -L_{16} & -L_{18} & 0 & L_{55} & L_{56} & L_{58} & L_{58} \\
L_2 &= 0 & -L_{16} & -L_{18} & 0 & L_{55} & L_{56} & -L_{L_{58}} & L_{58} \\
L_3 &= -L_{16} & -L_{14} & -L_{13} & -L_{13} & -L_{13} & -L_{13} & -L_{13} & -L_{13} \\
L_4 &= -L_{16} & -L_{14} & -L_{13} & L_{17} & L_{37} & L_{77} & L_{77} & L_{77} \\
L_5 &= 0 & -L_{16} & -L_{18} & 0 & L_{55} & L_{56} & L_{58} & L_{58} \\
L_6 &= 0 & -L_{16} & -L_{18} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[\mathcal{F}(\mathcal{T}) = \begin{bmatrix}
\frac{\nu h^3 (3\lambda + 5\mu)}{2(\lambda + 2\mu)} & \frac{\nu h^3 (3\lambda + 5\mu)}{2(\lambda + 2\mu)} & -\nu h^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}^T.
\]

The operators \(L_i\) are defined as follows:

\[L_{11} = c_2 - \alpha_{x_1}^2, \quad L_{12} = (1-c_2)-\alpha_{x_1}^2, \quad L_{13} = c_2 - \alpha_{x_2}^2, \quad L_{14} = \alpha_{x_1}^2, \quad L_{16} = \alpha_{x_2}^2, \quad L_{18} = \alpha_{x_1}^2, \quad L_{35} = \alpha_{x_2}^2, \quad L_{36} = \alpha_{x_1}^2, \quad L_{38} = \alpha_{x_2}^2, \quad L_{39} = \alpha_{x_1}^2, \quad L_{55} = \alpha_{x_2}^2, \quad L_{56} = \alpha_{x_1}^2, \quad L_{58} = \alpha_{x_2}^2, \quad L_{59} = \alpha_{x_1}^2.
\]

The parametric system (82) is an order two elliptic system of nine partial differential equations, where \(L\) is a linear differential operator acting on the set of kinematic variables \(\mathcal{T}\). The ellipticity of the system (82) can be easily verified by calculating the determinant of the principle symbol \(L(\xi)\)

\[\det L(\xi) = \frac{256 h^4}{375(\alpha + \mu)} c_1 c_2 c_3 c_4 (\xi_1^2 + \xi_2^2)^5,
\]

where \(c_1 = \xi_1, c_2 = \xi_2\) is an arbitrary vector from \(\Omega^2\). For nonzero elastic constants the determinant of \(L(\xi)\) is nonzero if and only if \(\xi \neq 0\), and thus the system of equations (82) is elliptic. The ellipticity condition for the system suggests the invertibility of the principal part of its symbol \(L(\xi)\) for all non-zero values of \(\xi \neq 0\). The right-hand side, and therefore the solution \(\mathcal{U}\) are the functions of the splitting parameter \(\gamma\). The optimal value of the parameter \(\gamma\) corresponds to the minimum of elastic energy over the micropolar strain field (55).

The system (82) should be complemented with the boundary conditions. We describe the case of the hard simply supported boundary conditions in the numerical analysis section.

2.3. Optimal value of the splitting parameter

As we have mentioned earlier, the equilibrium systems of partial differential equations correspond to a state of the system (82) where the minimum of the energy is reached. The optimization of the splitting parameter appears as a result of the Generalized Hellinger–Prange–Reissner (HPR) principle for the micropolar (64). The bending system of Eqs. (65)-(70) depends on the splitting parameter and therefore its solution is parametric. The minimization procedure for the elastic energy allows us to find the optimal value of this parameter, which corresponds to the unique solution of the bending problem.

Taking advantage of the linearity of the system (82) we consider two cases:

\[\mathcal{U}_0 = \mathcal{F}(0), \quad \mathcal{U}_1 = \mathcal{F}(1),
\]

(83)

where \(\mathcal{U}_0\) is a solution of the bending system of Eqs. (82) when \(\eta = 0\), and \(\mathcal{U}_1\) when \(\eta = 1\).
Let us define \( \mathcal{U}_t \) as a linear combination of \( \mathcal{U}_0 \) and \( \mathcal{U}_1 \):

\[
\mathcal{U}_t = (1-\eta)\mathcal{U}_0 + \eta \mathcal{U}_1.
\] (85)

Notice that since \( p_1 = \eta p, \ p_2 = \left( \frac{1}{2} \right) p, \ \mathcal{F}(\eta) = (1-\eta)\mathcal{F}(0) + \eta \mathcal{F}(1) \) and therefore due to the linearity of system (82)

\[
\mathcal{U}_t = \mathcal{L}((1-\eta)\mathcal{U}_0 + \eta \mathcal{U}_1) = (1-\eta)\mathcal{F}(0) + \eta \mathcal{F}(1) = \mathcal{F}(\eta).
\]

The set of kinematic variables \( \mathcal{U}_t \) defined as (85) is a solution of the micropolar plate bending system of Eqs. (82).

The optimal value of the splitting parameter \( \eta \) is a minimizer of the density of the work \( \mathcal{W}^{\eta} \) done by the stress and couple stress over the micropolar strain field. Let us obtain an explicit expression for the optimal value of the parameter \( \eta \).

Let us represent each component of the stress set \( \mathcal{S}^{\eta} \) and strain set \( \mathcal{E}^{\eta} \) as a linear combinations of the stress sets \( \mathcal{S}^{(0)} \) and \( \mathcal{S}^{(1)} \), and the strain sets \( \mathcal{E}^{(0)} \) and \( \mathcal{E}^{(1)} \) respectively

\[
\mathcal{S}^{\eta} = (1-\eta)\mathcal{S}^{(0)} + \eta \mathcal{S}^{(1)},
\]

\[
\mathcal{E}^{\eta} = (1-\eta)\mathcal{E}^{(0)} + \eta \mathcal{E}^{(1)}.
\]

Therefore

\[
\mathcal{W}^{\eta} = \mathcal{S}^{\eta} \cdot \mathcal{E}^{\eta} = ((1-\eta)\mathcal{S}^{(0)} + \eta \mathcal{S}^{(1)}) \cdot ((1-\eta)\mathcal{E}^{(0)} + \eta \mathcal{E}^{(1)}).
\]

The zero of the derivative \( \partial \mathcal{W}^{\eta}\!/\!\partial \eta \) gives the optimal value \( \eta_0 \) of the splitting parameter \( \eta \). Thus

\[
\eta_0 = \frac{2 \mathcal{W}^{(0)} - \mathcal{W}^{(1)}}{2 \mathcal{W}^{(1)} + \mathcal{W}^{(0)}},
\] (86)

and the optimal value is

\[
\eta_0 = \frac{2 \mathcal{W}^{(0)} - \mathcal{W}^{(1)}}{2 \mathcal{W}^{(1)} + \mathcal{W}^{(0)}},
\]

where

\[
\mathcal{W}^{(0)} = \mathcal{S}^{(0)} \cdot \mathcal{E}^{(0)},
\]

\[
\mathcal{W}^{(1)} = \mathcal{S}^{(1)} \cdot \mathcal{E}^{(1)},
\]

\[
\mathcal{W}^{(0)} = \mathcal{S}^{(0)} \cdot \mathcal{E}^{(0)},
\]

\[
\mathcal{W}^{(1)} = \mathcal{S}^{(1)} \cdot \mathcal{E}^{(1)}.
\]

3. Numerical analysis

In this section we describe the Finite Element algorithm used to solve the micropolar field equations. We develop the algorithm for the calculation of the optimal value of the splitting parameter. We compare the main kinematic variables for the square plate made of dense polyurethane foam with the analytical solution of the three-dimensional micropolar elasticity presented in [33] and provide the analysis of the plates of different shapes.

3.1. Comparison of the analytical solutions

In our computations we consider plates made of polyurethane foam—a material reported in the literature to be micropolar [21]. Insulation materials made from rigid polyurethane foam are used in the construction of large industrial and agricultural buildings. It is known that in the plaster supports in wooden structures and bridging large spans between the top chords in flat roofing, the insulation materials are exposed to bending stresses [15].

Therefore the prediction of the behavior of the polyurethane foam under bending stress is extremely important. Another example of the polyurethane foam plates can be found in the structural insulated panels—a high-performance walling materials, where the sides of the panel are made of cement, plywood, or oriented strand board, and the middle part is made of highly dense polyurethane foam. The structural insulated panels are widely used in walls, floor slabs and roofs of houses and commercial buildings.

In our calculations we will use the following technical parameters: Young's modulus \( E \), Poisson's ratio \( \nu \), the characteristic length for bending \( l_b \), the characteristic length for torsion \( l_t \), the coupling number \( N \).

The conversion formulas between the technical constants and elastic parameters [17]

\[
E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)},
\]

\[
l_b = \frac{1}{2} \sqrt{\frac{\nu+\epsilon}{\mu}}, \quad l_t = \frac{\nu}{\mu}, \quad N = \sqrt{\frac{\mu}{\mu+\alpha}},
\]

imply the following inverse conversion formulas:

\[
\lambda = \frac{E_v}{2(\nu+1)}, \quad \mu = \frac{E}{2(\nu+1)}, \quad \alpha = \frac{E N^2}{2(\nu+1)},
\]

\[
\beta = \frac{E_l}{2(\nu+1)}, \quad \gamma = \frac{E l_t^2}{2(\nu+1)}, \quad \epsilon = \frac{E(4l_t^2-l_j^2)}{2(\nu+1)}.
\]

Let us consider a plate of thickness \( h = 0.1 \text{ m} \) made of polyurethane foam, and the ratio \( a/h \) varying from 5 to 30. The values of the technical elastic parameters for the polyurethane foam are reported in [21]: \( E = 299.5 \text{ MPa}, \quad \nu = 0.44, \quad h_1 = 0.62 \text{ mm}, \quad h_b = 0.327, \quad N^2 = 0.04 \). These values correspond to the following values of Lamé and asymmetric parameters: \( \mu = 103.993 \text{ MPa}, \quad \lambda = 4.333 \text{ MPa}, \quad \beta = 39.975 \text{ MPa}, \quad \gamma = 39.975 \text{ MPa}, \quad \epsilon = 4.505 \text{ MPa} \) (the ratio \( \beta/\gamma \) is equal to 1 for the bending).

Let us first analyze the case of the square plate and compare the analytical solution for the field equations (82) and the analytical solution of the corresponding three-dimensional micropolar elasticity problem presented in [33]. Let us consider the following hard simply supported boundary conditions:

\[
W = 0, \quad W^s = 0, \quad W^r \cdot s = 0, \quad n \cdot M_n = 0, \quad S^r \cdot n = 0, \quad Q^r \cdot s = 0, \quad \mathbf{Q} \cdot \mathbf{S} = 0, \quad \mathbf{S} \cdot R_n = 0, \quad \mathbf{S} \cdot \mathbf{R}^* \mathbf{n} = 0, \quad (91)
\]

where \( \mathbf{n} \) and \( \mathbf{s} \) are the normal and the tangent vectors to the boundary \( \partial \mathcal{B} \), respectively.

Let the body \( B_0 \) be a square plate \( a \times a \) of thickness \( h \): \( B_0 = [0,a] \times [0,a] \times [-h/2,h/2] \). The boundary of the body is given by

\[
G = G_1 \cup G_2
\]

where

\[
G_1 = \{(x_1, x_2) : x_1 \in \mathcal{I}, \ x_2 \in [0,a] \},
\]

\[
G_2 = \{(x_1, x_2) : x_1 \in [0,a], \ x_2 \in \mathcal{I} \},
\]

and \( l = \{0,a\} \) is the set of two numbers: 0 and \( a \).

For the square plate, the hard simply supported boundary conditions (90) and (91) can be represented in the following mixed Dirichlet–Neumann form:

\[
G_1 : \mathbf{W} = 0, \quad W^s = 0, \quad W^r = 0, \quad \mathbf{L}_2^0 = 0, \quad \mathbf{L}_1^0 = 0,
\]

\[
G_2 : \mathbf{W} = 0, \quad W^s = 0, \quad W^r = 0, \quad \mathbf{L}_2^0 = 0, \quad \mathbf{L}_1^0 = 0,
\]

\[
(92)
\]
The analytical solution of the three-dimensional micropolar bending problem is presented in [33]. The two-dimensional field equations (82) can be solved by applying the method of separation of variables. We obtain the kinematic variables in the following equations for \( \Psi \) in MPa units:

\[
\Psi_1 = A_1 \cos \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right), \\
\Psi_2 = A_2 \sin \left( \frac{\pi x_1}{a} \right) \cos \left( \frac{\pi x_2}{a} \right), \\
W = A_3 \sin \left( \frac{\pi x_1}{a} \right) \cos \left( \frac{\pi x_2}{a} \right), \\
\Omega^2_1 = A_4 \cos \left( \frac{\pi x_1}{a} \right) \cos \left( \frac{\pi x_2}{a} \right), \\
\Omega^2_2 = A_5 \cos \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right), \\
\Omega^0_2 = A_6 \cos \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right), \\
W^r = A_7 \sin \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right), \\
\Omega^0_1 = A_8 \cos \left( \frac{\pi x_1}{a} \right) \cos \left( \frac{\pi x_2}{a} \right), \\
\Omega^0_2 = A_9 \cos \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right).
\]  

We find \( A_i \) by substituting the expressions (96) into the system of Eqs. (82) and solving the obtained system of nine linear equations for \( A_i \).

The comparison of the analytical solutions for the maximum vertical deflection and the maximum microrotation of the two-dimensional model versus the analytical solutions for three-dimensional micropolar elasticity is provided in Tables 1 and 2.

The comparison of the numerical values of the kinematic variables for the square plate made of dense polyurethane foam with the analytical solution of the three-dimensional micropolar elasticity confirms the high order of approximation of three-dimensional (exact) solution. The relative error of order 1% for the maximum vertical deflection is compatible with the precision of the Reissner plate theory [10] (Fig. 1).

The comparison of the transverse variations of the components of the displacement of the two-dimensional micropolar plate theory versus three-dimensional micropolar elasticity theory is given in Fig. 2.

### 3.2. Finite element analysis

Let us look for the solution of the micropolar plate bending system of Eqs. (82) in the Hilbert space \( V \) defined as

\[ V = V_1 \times V_2 \times V_3 \times V_4 \times V_5 \times V_6 \times V_7 \times V_8 \times V_9, \]

where \( V_1 = V_6 = V_9 = (v \in H^1(B_0), v = 0 \text{ on } G_1), \quad V_2 = V_5 = V_8 = (v \in H^1(B_0), v = 0 \text{ on } G_2), \quad V_3 = V_7 = (v \in H^1(B_0), v = 0 \text{ on } G), \]

\( V_4 = H^1(B_0) \), and \( H^n(B_0) \) is a standard Hilbert space of functions that are square-integrable together with their first partial derivatives.

The weak formulation of the micropolar plate bending problem is obtained as follows. Let \( v \in H^1(B_0) \). By multiplying both sides of (82) by \( \Psi \) we have

\[ \Psi LL = \Psi F(\eta) \]

and then integrating over the plate \( B_0 \) we obtain

\[ \int_{B_0} \Psi LL \, ds = \int_{B_0} \Psi F(\eta) \, ds. \]

Let us introduce the following notation:

\[ a(\xi, \eta) = \int_{B_0} \Psi L \xi L \eta \, ds, \]

\[ f(\eta) = \int_{B_0} \Psi F(\eta) \, ds. \]

The corresponding weak problem is formulated as follows: find all \( \Psi \in V \) such that for all \( \Psi \in H^1(B_0) \)

\[ a(\xi, \eta) = f(\eta). \]  \( \text{(99)} \)

Note that the form \( a(\xi, \eta) = \int_{B_0} \Psi L \xi L \eta \, ds \) is a summation of the weak forms of the operators \( L_i \). In order to obtain the expression for the weak formulation (99) we observe that there are only three types of linear operators present in the expression for \( a(\xi, \eta) \)—operators of order one, zero and two:

\[ I^{(0)} = 1. \]  \( \text{(100)} \)

### Table 1
Comparison of the maximum vertical deflection \( u_1 \) of the polyurethane foam square plate.

<table>
<thead>
<tr>
<th>( a/h )</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal value ( u_1 )</td>
<td>0.2383156</td>
<td>0.1280473</td>
<td>0.0795667</td>
<td>0.0548782</td>
<td>0.0407998</td>
</tr>
<tr>
<td>Micropolar plate model (m)</td>
<td>0.0236306</td>
<td>0.0622810</td>
<td>0.1205956</td>
<td>0.2011537</td>
<td>0.3076744</td>
</tr>
<tr>
<td>Micropolar 3D elasticity (m)</td>
<td>0.0237443</td>
<td>0.0624282</td>
<td>0.1207626</td>
<td>0.2013360</td>
<td>0.3078709</td>
</tr>
<tr>
<td>Relative error (%)</td>
<td>0.47</td>
<td>0.23</td>
<td>0.13</td>
<td>0.09</td>
<td>0.06</td>
</tr>
</tbody>
</table>

### Table 2
Comparison of the maximum microrotation \( \phi_1 \) of the polyurethane foam square plate.

<table>
<thead>
<tr>
<th>( a/h )</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal value ( \phi_1 )</td>
<td>0.2383156</td>
<td>0.1280473</td>
<td>0.0795667</td>
<td>0.0548782</td>
<td>0.0407998</td>
</tr>
<tr>
<td>Micropolar Plate Model</td>
<td>0.0013365</td>
<td>0.0052364</td>
<td>0.0132366</td>
<td>0.0266257</td>
<td>0.0467707</td>
</tr>
<tr>
<td>Micropolar 3D Elasticity</td>
<td>0.0013430</td>
<td>0.0052686</td>
<td>0.0132545</td>
<td>0.0266490</td>
<td>0.0467995</td>
</tr>
<tr>
<td>Relative Error (%)</td>
<td>0.48</td>
<td>0.23</td>
<td>0.14</td>
<td>0.09</td>
<td>0.06</td>
</tr>
</tbody>
</table>
The weak forms of these operators are given as follows:

\[
L^{(1)} = \frac{\partial}{\partial x_i} \\
L^{(2)} = \nabla \cdot AV .
\]

where \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \).

\[
\int_{S_0} v(L^{(1)}u) \, ds = c \int_{S_0} vu \, da.
\]
\[
\int_\Omega (1) u \, ds = \int_\Omega \frac{\partial u}{\partial x_i} \, da,
\]

\[
\int_\Omega (2) u \, ds = \int_\Omega \nabla u : \nabla v \, da - \int_\Omega (\nabla u : n) v \, dr - \int_\partial \Omega A v n \, da,
\]

where the weak form of the second order operator is obtained by performing the corresponding integration by parts.

Note that all second order operators in the weak formulation can be represented in the form

\[
\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \nabla \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla,
\]

\[
\frac{\partial^2}{\partial x_1 \partial x_2} = \nabla \cdot \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \nabla,
\]

\[
c_1 \frac{\partial^2}{\partial x_1^2} + c_2 \frac{\partial^2}{\partial x_2^2} = \nabla \cdot \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \nabla.
\]

In the previous section we showed that the system (82) represents an elliptic system of partial differential equations. It is possible to prove that the operator \(L\) is in fact strongly elliptic. This will imply the coerciveness of the operator \(L\) and its bilinear form on \(V\), which is important for the verification of the hypothesis of Babuška–Lax–Milgram theorem (see [25,3] for details). The corresponding convergence analysis of finite element approximations of the elliptic systems and the analytic estimates on the domains with corners are given in [8].

**Theorem 1** (Strong ellipticity of the bending system of equations). The micropolar plate bending system of Eqs. (82) is strongly elliptic.

**Proof.** Let \(\xi = (\xi_1, \xi_2)\) and \(v = (v_1, v_2, v_3, \nu_z, v_{\nu_z})\) be some arbitrary vectors from \(\mathbb{R}^2\) and \(\mathbb{R}^9\) respectively. Let us assume that \(\xi = (\xi_1, \xi_2) \neq 0\) and denote the principle symbol of the system of equations (82) by \(l(\xi)\). The bilinear form \(a(\xi)v^T\) can be expressed as

\[
o a(\xi)v^T = I_1 + I_2 + I_3 + I_4 + I_5,
\]

where

\[
I_1 = c_1(v_1 \xi_1 + v_2 \xi_2)^2 + c_2(v_2 \xi_1 - v_1 \xi_2)^2,
\]

\[
I_2 = c_4 \xi_1^2 (\xi_1^2 + \xi_2^2),
\]

\[
I_3 = c_4 \xi_2^2 (\xi_1^2 + \xi_2^2)^2 + \frac{4}{\delta} c_1 \xi_2^2 (\xi_1^2 + \xi_2^2),
\]

\[
I_4 = c_4 (2 v_3 \xi_1 + v_{\nu_z} \xi_2)^2 + (v_{\nu_z} \xi_1 - v_3 \xi_2) (v_{\nu_z} \xi_1 + v_{\nu_z} \xi_2) + (v_{\nu_z} \xi_1 + v_{\nu_z} \xi_2)^2,
\]

\[
I_5 = c_4 (2 v_3 \xi_1 - v_{\nu_z} \xi_2)^2 + (v_{\nu_z} \xi_1 - v_3 \xi_2) (v_{\nu_z} \xi_1 - v_{\nu_z} \xi_2) + (v_{\nu_z} \xi_1 - v_{\nu_z} \xi_2)^2.
\]

It is easy to check that \(I_1, I_2, I_3\) and \(I_5\) are always nonnegative. Indeed, since \(c_1, c_2 > 0\) then \(I_1 > 0\) for all \(\xi \in \mathbb{R}^2\), and \(I_1 = 0\) if and only if \(v_1 = v_2 = 0\). Since \(c_4 > 0\) then \(I_2 \geq 0\) for all \(\xi \in \mathbb{R}^2\), and \(I_2 = 0\) if and only if \(v_3 = 0\). Since \(c_4 > 0\) then \(I_3 \geq 0\) for all \(\xi \in \mathbb{R}^2\), and \(I_5 = 0\) if

---

**Fig. 3.** The cross section that contains the center of the micropolar square plate 2.0 m × 2.0 m × 0.1 m made of polyurethane foam without load (left) and under the load (right). The applied sinusoidal load is represented by the dashed black line.
and only if $\nu_3 = \nu_7 = 0$. The analysis of $l_4$ and $l_5$ is very similar since they have quadratic form: $\frac{1}{2}x^2 + xy + y^2$ with non-positive discriminant. Since $c_7 > 0$ and $c_8 > 0$ then $l_4 \geq 0$ and $l_5 \geq 0$; $l_3 = 0$ if and only if $\nu_8 = \nu_9 = \nu_5 = \nu_6 = 0$, and $l_4 = 0$ if and only if $\nu_8 = \nu_9 = \nu_5 = \nu_6 = 0$. Since $l_7 \geq 0$, then $aU^T \xi U = 0$ if and only if all $l_i = 0$. The last is equivalent to $v = 0$ and therefore we conclude that the operator $L$ is strongly elliptic.

The strong ellipticity of the operator $L$ implies the coerciveness of the bilinear form $A(U, V)$ and therefore by Babuška-Lax-Milgram theorem there exists a unique solution $U \in V$ to the weak (99).

The proposed minimization of the plate stress energy allows us to introduce the following effective algorithm for the optimal value of the splitting parameter $\eta$, which can be used for the plate of an arbitrary shape.

![Fig. 4. Hard simply supported square plate 2.0 m x 2.0 m x 0.1 m made of polyurethane foam: the initial mesh and the isometric view of the resulting vertical deflection of the plate.](image)

![Fig. 5. Nonrectangular hard simply supported plate made of polyurethane foam: (a) the area where the sinusoidal load is applied, (b) initial mesh, (c) the density plot and (d) the isometric view of the resulting vertical deflection of the plate.](image)
Algorithm 2 (Optimal value of the splitting parameter).
1. Solve the systems (82) and (83) for \( \nu_0 \) and \( \nu_1 \) respectively.
2. Calculate the components of the micropolar plate stress sets \( S_0 \) and \( S_1 \) from the sets of kinematic variables \( \nu_0 \) and \( \nu_1 \), using the constitutive formulas in the reverse form (71)–(80).
3. Calculate the components of the micropolar plate strain sets, \( \varepsilon_0 \) and \( \varepsilon_1 \) from the sets of kinematic variables \( \nu_0 \) and \( \nu_1 \), using the strain-displacement relations (57)–(63).
4. Find the work densities \( W_{00}, W_{11}, W_{10} \) and \( W_{01} \) by substituting the micropolar plate stress and strain sets \( S_0, S_1, \varepsilon_0 \) and \( \varepsilon_1 \) into the definitions (86)–(89).
5. Substitute the values of the work densities \( W_{00}, W_{11}, W_{10} \) and \( W_{01} \) into the expression for the optimal value of the splitting parameter \( \eta_0 \) (86).

The above algorithm can be efficiently used to find the solution of the system of equations (82). Indeed, once the optimal value of the splitting parameter \( \eta_0 \) is found, the solution \( \nu_{00} \) of the bending system of Eqs. (82) is found from (85) as a linear combination of the sets of kinematic variables \( \nu_0 \) and \( \nu_1 \)

\[
\nu_{00} = (1 - \eta_0 k_0 + \eta_0) \nu_1.
\]

The numerical modeling of the vertical deflection of the square plate made of polyurethane foam is given in Figs. 3 and 4.

The distribution of the vertical deflection for nonrectangular plates and plates with rectangular holes is given in Figs. 5–7.

The effect of the asymmetric part on the maximal vertical deflection of the plate made of polyurethane foam is shown in Table 3 and Fig. 8 for \( a = 1.0 \, \text{m} \) and the ratio \( a/h \) varying from 10 to 30.

Fig. 8 provides information for comparison between micropolar and classical elasticity. The energy of microrotation becomes a part of the total elastic energy. This causes the redistribution of the elastic energy depending on the value of the asymmetric part. We perform computations for different levels of the asymmetric microstructure by reducing the values of the elastic asymmetric parameters. In the case when 0.1% of real values of asymmetric parameters are used, i.e. the microstructure is almost irrelevant, the solution of the proposed model converges to the corresponding Reissner solution. Fig. 8 illustrates the size effect of micropolar plate theory which predicts that plates of smaller thickness will be more rigid than would be expected on the basis of the Reissner plate theory. Similar experimental behavior was reported in [21] for torsion and bending of cylindrical rods of a Cosserat solid.

4. Conclusion

In this paper we presented the Finite Element modeling of the bending of micropolar elastic plates. Based on our enhanced mathematical model for Cosserat plate bending [33] we introduced the micropolar plate field equations as an elliptic system of nine differential equations in terms of the kinematic variables. We have developed the efficient algorithm for finding the optimal value of the splitting parameter and discussed the approximations of stress and couple stress components. The numerical algorithm includes the efficient method for finding the solution of the micropolar plate field

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equations corresponding to the optimal value of the splitting parameter. The comparison of the numerical values of the kinematic variables for the square plate made of the dense polyurethane foam with the analytical solution of the three-dimensional micropolar elasticity confirmed the high order of approximation of three-dimensional solution. The size effect of micropolar plate theory predicts that plates made of smaller thickness will be more rigid than would be expected on the basis of the Reissner plate theory. The numerical modeling of the bending nonrectangular plates and plates with rectangular holes made of the polyurethane foam under different loads have been provided.

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References